MAC-CPTM Situations Project

Situation 04: Bull's Eye!

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Prompt

In prior lessons, students learned to compute mean, mode and median. The teacher presented the formula for standard deviation and had students work through an example of computing standard deviation with data from a summer job context. The following written work developed during the example:

x	$\bar{x}$	$(x - \bar{x})$	$(x - \bar{x})^2$
140	200	-60	3600
190	200	-10	100
210	200	10	100
260	200	60	3600

$$\sum (x_i - \bar{x})^2 = 7400$$

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

$$\sigma = \sqrt{\frac{\sum \left(x - \bar{x}\right)^2}{n}} \qquad \sigma = \sqrt{1850}$$

$$\sigma = \sqrt{\frac{7,400}{4}} \qquad \sigma \approx 43.01$$

The teacher then said, "Standard deviation is a measure of the consistency of our data set. Do you know what consistency means?" To explain "consistency" the teacher used the idea of throwing darts. One student pursued the analogy, "If you hit the bull's eye your standard deviation would be lower. But if you're all over the board, your standard deviation would be higher." The student drew the following picture to illustrate his idea:
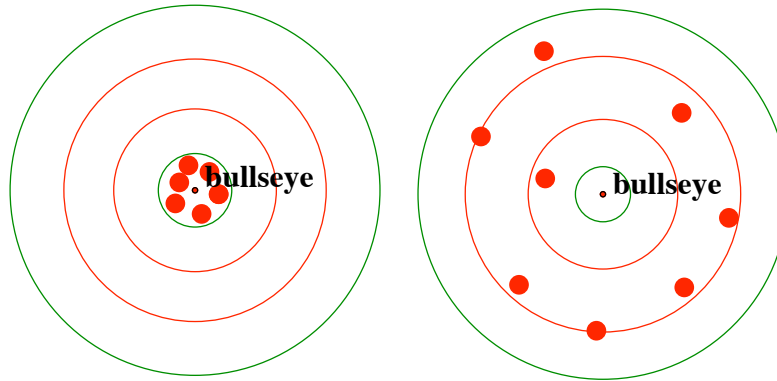
A student raised her hand and asked, "But what does this tell us about what we are trying to find?"

Commentary

This prompt seems to deal with the analogy of the data points, mean, and the standard deviation of a data set as a dartboard target and the places on the target where the darts landed. Two key aspects of the visual representation are the objects and the relative positions of those objects. The robustness of the representation depends upon how well these two aspects collectively represent the key mathematical objects being represented (e.g., data points, mean, standard deviation) and the relationships among those objects. These two aspects apply to familiar representations (e.g. box and whiskers plots, formulas) as well as novel representations (e.g. bull's eye target).

Mathematical Foci

Mathematical Focus 1
If elements of a visual depiction of a formula represent symbolic elements of the formula, the relative positions of those elements should be consistent with the meaning of the symbols.

In the formula, we have σ, x, $\bar{x}$, and n. The number of darts is n and in analogous to the number of data points in the data set. Every data point or each value of x is represented by the place a dart lands. For the two targets shown in the prompt, the bull's eye seemingly is used in lieu of the mean, $\bar{x}$. Each distance between a dart and the bull's eye is associated with a value of $\left|(x - \bar{x})\right|$. No one part of the target represents σ, but a greater spread of the darts suggests a greater value for σ.

Mathematical Focus 2
A robust representation of a data set and its mean and standard deviation allows for the mean and standard deviation to be defined by that data set and for the representation to reflect the relationship between mean and standard deviation.
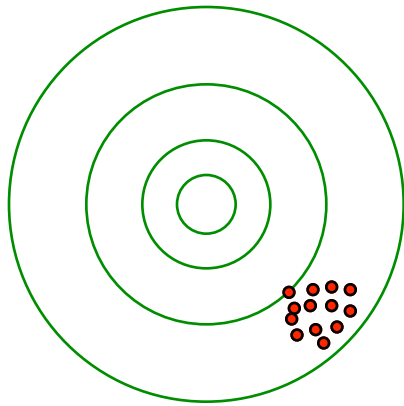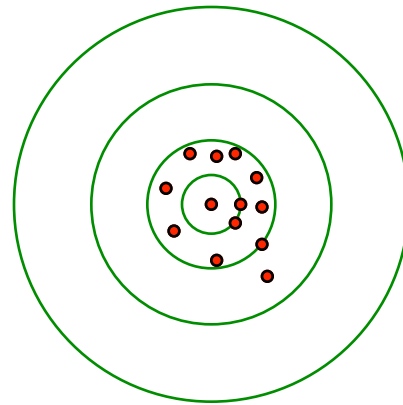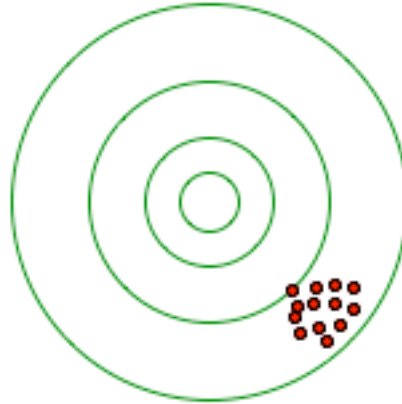
Target A   **Target B**

Targets A and B, shown in this Focus, represent different sets of data. In Target A, unlike in Target B, the bull's eye (center point of a target) does not represent the mean of the set of data points. Target A represents a data set with a smaller standard deviation than the data set represented in Target B, in spite of the fact that none of the hits in Target A are near the bull's eye. This is because the data points represented on Target B are more spread out than those of Target A. These diagrams can be confusing if we do not keep in mind that the target is an analogy, and as a representation it does not capture the notion of mean as the mean position of the data points.

There may be other interpretations of the diagram. For example, the target may represent a population, the bull's eye may represent the population mean, and the hits may represent a sample from that population. With this interpretation, Target B represents a data set with a sample mean close to the population mean and Target A represents a data set with a sample mean that is not close to the population mean.

Mathematical Focus 3
Different representations might underscore different features of the mathematics they represent such as those that emphasize either magnitude or sign of numbers.

In the target diagram, the bull's eye is a predetermined location for the mean. However, the mean of a data set depends on the data set. It is possible to represent a set of data for which the mean would not be the bull's eye, as in Target A.

Target A

A better representation would have the mean depend upon the data set and not be a fixed element of the representation. In a different representation, the value of x is could be based on its shortest distance to the bull's eye and the sample mean, $\bar{x}$, would be the mean distance that data points are away from the bull's eye. To represent the mean, we draw a circle with the center remaining at the bull's eye, but with a radius of $\bar{x}$. In this new representation, the sign of $(x - \bar{x})$ relates to whether the data point is inside or outside this new circle. Any data point that lies on the interior of this newly constructed circle will have a distance from the bull's eye that is less than the radius of the new circle. As a result, the value of $(x - \bar{x})$ will be negative. For data point that is on the exterior of the new circle, the value for $(x - \bar{x})$ will be positive.

Additionally, from this new representation, we can create a one-dimensional representation of the data. For each data point, construct the circle through the data point with the bull's eye as its center. Construct one ray from the bull's eye. The points of intersection of this ray with the circles for the data points and the mean form the one-dimensional representation of the data. This one dimensional representation shows that while distance is always positive, the value of $(x - \bar{x})$ can be positive, negative, or zero.
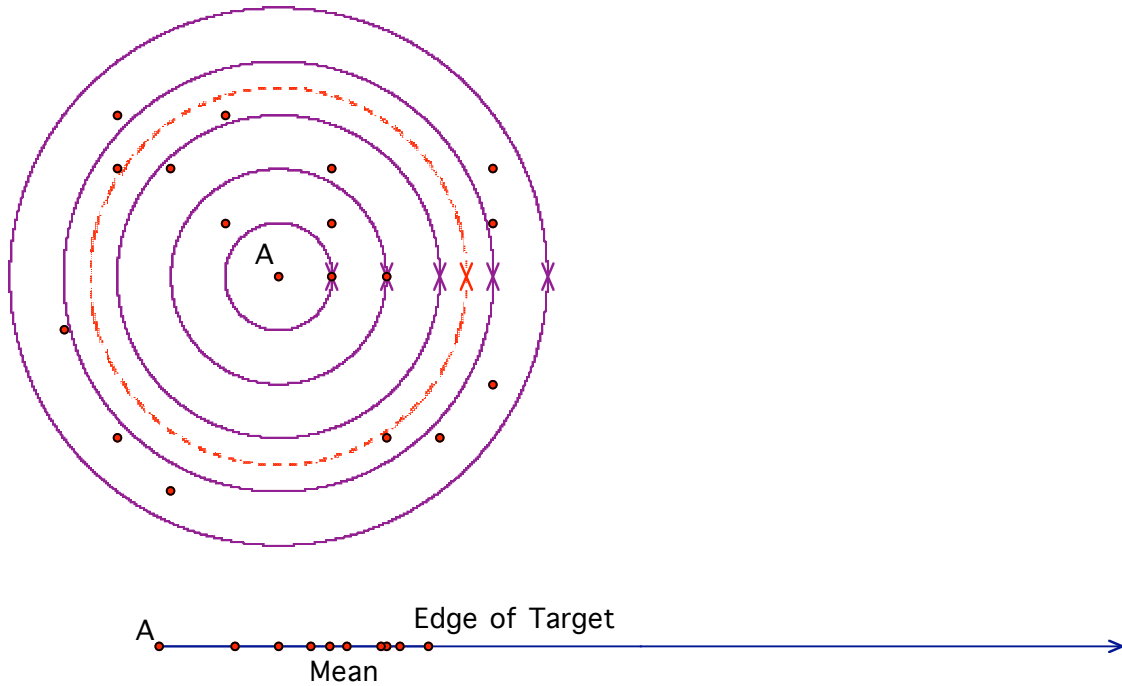
It is important to remember that this transformation assumes that all of the data points are non-negative. If the data set contains negative values, this problem can be remedied by a transformation of the data that adds the absolute value of the least data value to each data value.

Mathematical Focus 4
The mean and median are measures of central tendency and the standard deviation is a measure of spread; different representations may be more or less effective in representing measures of these two different types.

The target diagram focuses on the mean as the "center" of the data set much like the box-and-whisker plot focuses on the median as the "center" of the data set. Both the target diagram and the box-and-whisker plot allow us to see something about the spread of the data. When the bull's eye represents the sample mean, the target diagram allows us to "see" something about the standard deviation in relation to the mean. We cannot use the box-and-whisker plot to "see" the standard deviation in relationship to the mean but we can see something about the spread of the data with respect to the median.